Lower Bound for the Leading Regge Trajectory*

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A lower bound is given for the high-energy limit of the invariant scattering amplitude $F(s,t)$ which implies that $\lim_{t\to\infty}|F(s,t)|\geq O(t^{-1})$ for all values of the momentum transfer variable s. The derivation of this bound is based upon some general notions of dispersion theory. The situation in potential scattering is discussed briefly.

RECENT experiments on the diffraction peak in high-energy p-p scattering[1] give us information on the high-energy limits of the scattering amplitude as a function of the momentum transfer variable. Therefore, it is of great interest to obtain theoretical predictions for the asymptotic behavior of scattering amplitudes on the basis of the general notions of relativistic dispersion theory. Using unitarity and double dispersion relations, Froissart[2] has derived an ***upper*** bound for the physical amplitude; considering the elastic scattering of spinless particles he finds $\lim_{t\to\infty}|F(s,t)|\leq Ct(\ln t)^2$ for $s\leq 0$, where $F(s,t)$ is the invariant amplitude, $t^{1/2}$ is the total energy, and $-s$ the square of the momentum transfer in the center-of-mass system.

It is the purpose of this note to give a corresponding *lower* bound for the high-energy limit of the amplitude. We show that

$$\lim_{t\to\infty}|F(s,t)|\geq O(t^{-1}) \quad \text{for} \quad s\leq 0.$$

Our proof is based on the general notions of dispersion theory. If there exists a leading Regge trajectory[3] $\lambda=\alpha(s)$ such that $F(s,t)\to B(s)t^{\alpha(s)}$ for $t\to\infty$, then our result implies that $\mathrm{Re}\,\alpha(s)\geq -1$ for real $s\leq 0$. If applied to p-p scattering, our limit is in agreement with present experiments[1] which cover the region $-5(\mathrm{BeV}/c)^2\lesssim s\leq 0$; however, the experimental errors are still very large and further experiments are certainly of greatest interest.

In order to have a simple example, we consider elastic pion-pion scattering and ignore isotopic spin variables. Then we have one invariant amplitude $F(s,t)$ which is a real analytic function and which is regular except for the usual branch points. We assume that $F(s,t)$ is bounded by a polynominal in t. Then there is a positive integer N such that the function[4]

$$F_\pm(s,\lambda)=\frac{1}{\pi}\int_{4\mu^2}^{\infty}dv\,\frac{1}{2q^2}Q_\lambda\left(1+\frac{v}{2q^2}\right)A_\pm(s,v), \tag{1}$$

with $4q^2=s-4\mu^2$, is regular for $\mathrm{Re}\lambda>N$ and defines there a unique interpolation of the partial wave amplitudes $F_l(s)$, $l=$even/odd. We have

$$A_\pm(s,v)=A_t(s,v)\pm A_u(s,v), \tag{2}$$

and the asymptotic behavior of the absorptive parts A_t and A_u determines the singularities of $F_\pm(s,\lambda)$ for $\mathrm{Re}\lambda\leq N$. At first, we assume that there are only isolated singularities so that we do not encounter a natural boundary; i.e., a line of singularities which completely prevents the continuation of $F_\pm(s,\lambda)$ to the left half of the λ plane. On the basis of these assumptions, it has been shown by one of us (R.O.)[5] that $F_\pm(s,\lambda)$ cannot have s-independent poles for $\mathrm{Re}\lambda\leq N$. Let us now assume that

$$|A_\pm(s,v)|\leq O(v^{-1-\epsilon}),\quad \epsilon>0 \quad \text{for} \quad v\to\infty. \tag{3}$$

Then the representation (1) for $F_\pm(s,\lambda)$ yields an analytic function for $\mathrm{Re}\lambda>-1-\epsilon$. The Legendre function $Q_\lambda(z)$ has, single poles at negative integer values of λ; we have

$$\lim_{\lambda\to-n}Q_\lambda(z)=\frac{1}{\lambda+n}P_{n-1}(z), \tag{4}$$

and the function $F_\pm(s,\lambda)$ could have a pole at $\lambda=-1$, such that

$$\lim_{\lambda\to-1}F_\pm(s,\lambda)=\frac{1}{\lambda+1}\frac{1}{2\pi q^2}\int_{4\mu^2}^{\infty}dv\,A_\pm(s,v). \tag{5}$$

However, according to the results of reference 5, we know that $F_\pm(s,\lambda)$ cannot have an s-independent pole like the one at $\lambda=-1$. Hence, either the residue

$$R_\pm(s)=R_t(s)\pm R_u(s)=\int_{4\mu^2}^{\infty}dv\,A_\pm(s,v) \tag{6}$$

vanishes identically, or the bound (3) is not valid and we have

$$|A_\pm(s,v)|\geq O(v^{-1}) \quad \text{for} \quad v\to\infty. \tag{7}$$

The absorptive parts $A_t(s,t)$ and $A_u(s,u)$ are analytic functions of s which are regular on a cut plane for all

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$t,u \geq 4\mu^2$. Because of the assumption (3), the residuum (6) has corresponding analytic properties. It must be identically zero if it vanishes for all s in some small neighborhood. But we cannot have $R_t(s) \equiv 0$ and/or $R_u(s) \equiv 0$ if the reaction in the t and/or the u channel is elastic. Suppose we have elastic scattering in the t channel. Then the optical theorem is applicable, and for $t \geq 4\mu^2$ we find

$$A_t(0,t) = [t(t-4\mu^2)]^{1/2}\sigma^{\text{tot}}(t) > 0, \tag{8}$$

which implies that $R_t(s) \not\equiv 0$ and, because of (7), $|A_t(s,t)| \geq O(t^{-1})$ for $t \rightarrow \infty$. Thus, we conclude that

$$\lim_{t\to\infty} |F(s,t)| \geq O(t^{-1}); \tag{9}$$

for real $s \leq 0$, this is the result which we have mentioned in the introduction. So far we have assumed that there is no natural boundary which prevents the analytic continuation into the neighborhood of $\lambda = -1$; but if there should be such a singularity, then our bound (9) holds trivially.

If the t channel is elastic, we can assume that the s channel has the quantum numbers of the vacuum so that $F_+(s,\lambda)$ has a pole $\lambda = \alpha_0(s)$ corresponding to the leading vacuum trajectory. We have $A_+(s,v) = O(v^{\alpha_0(s)})$ for $v \rightarrow \infty$ and the representation (1) defines a regular function for $\text{Re}\lambda > \text{Re}\alpha_0(s)$. Our bound (9) implies that

$$\text{Re}\alpha_0(s) \geq -1. \tag{10}$$

Especially, it follows that for physical values of the momentum transfer s we have $\lim_{s\to-\infty}\alpha_0(s) \geq -1$.[6] Once we adopt the notion of a leading vacuum trajectory, our bound (9) has been automatically generalized to all reactions for which this Regge pole determines the asymptotic behavior, whether they are elastic or not. The inequalities (9) or (10) are mainly of interest in the physical region of the momentum transfer variable s, but they are also applicable for other points. A better lower bound for the amplitude is available for s near a two-particle threshold, like $s = 4\mu^2$ in our example. It follows from the representation (1) and the unitarity condition that for $s \rightarrow 4\mu^2$ there must be at least one pole of $F_\pm(s,\lambda)$ for $\text{Re}\lambda \geq -\frac{1}{2}$.[7]

We would like to add a few remarks concerning the validity of our arguments within the framework of potential scattering. We consider only such superpositions of Yukawa potentials for which a dispersion relation is valid[8] so that we can write down a representation corresponding to Eq. (1):

$$F(\nu,\lambda) = \frac{1}{\pi}\int_{\mu_0^2}^{\infty} dt\, Q_\lambda\left(1+\frac{t}{2\nu}\right)\frac{1}{2\nu}A_t(\nu,t), \tag{11}$$

where

$$A_t(\nu,t) = -\rho(t) + \frac{1}{\pi}\int_0^{\infty} d\nu' \frac{\rho(\nu',t)}{\nu'-\nu}, \tag{12}$$

and

$$V(r) = \int_{\mu_0^2}^{\infty} dt\, \rho(t)\exp(-t^{1/2}r). \tag{13}$$

If $V(r)$ has a r^{-1} singularity for $r \rightarrow 0$, i.e., if

$$\int_{\mu_0^2}^{\infty} dt\, \rho(t) \neq 0,$$

then the residue $R(\nu)$ corresponding to Eq. (6) cannot be identical zero, because we have $R(-\infty) \neq 0$, and hence, there must be at least one Regge pole with $\text{Re}\alpha(\nu) \geq -1$. This is a well-known result for Yukawa potentials,[9] because there is always a trajectory with $\alpha(-\infty) = -1$ and $d\alpha(\nu)/d\nu \geq 0$, for real $\nu < 0$. But what happens if $V(r)$ is regular at $r=0$ so that

$$\int_{\mu_0^2}^{\infty} dt\, \rho(t) = 0\,?$$

Then $R(-\infty) = 0$, and since the amplitude $F(\nu,t)$ approaches the Born approximation for $\nu \rightarrow -\infty$, it can happen that

$$\lim_{t\to\infty}\lim_{\nu\to-\infty} A_t(\nu,t) = O(t^{-2}).$$

Since we have no crossed channels in potential scattering, the residue $R(\nu)$ of the pole at $\lambda = -1$ could vanish identically. However, if it does not vanish, then there must be a Regge pole with $\text{Re}\alpha(\nu) \geq -1$ such that its residue vanishes for $\nu \rightarrow -\infty$.

Note added in proof. After this paper had been submitted for publication, we saw a Letter by V. N. Gribov and I. Ya. Pomeranchuk [Phys. Letters **2**, 239 (1962)] in which a result analogous to ours has been obtained. However, these authors give a complicated proof based upon a term in the discontinuity of $F(s,\lambda)$ involving the double spectral function $\rho_{ut}(u,t)$, which is not present in most approximation schemes. Our derivation uses only simple crossing, and it does not even make actual use of the full double dispersion relation. We believe that our approach is sufficiently more direct and general than that of Gribov and Pomeranchuk to warrant its publication.

In order to avoid any misunderstanding, we would also like to point out that the leading Regge trajectory is always the one with the largest value of the real part. If there is an intersection of two or more trajectories, then one simply has to follow the leading branch.

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[6] For real $s < 4\mu^2$, the trajectory function $\alpha_0(s)$ is generally real except for cuts due to possible intersections with other trajectories. That $\alpha_0(s)$ has otherwise only the right-hand cuts has been proven by R. Oehme and G. Tiktopoulos, Phys. Letters **2**, 86 (1962). For corresponding proofs in the strip approximation or in potential scattering, see A. O. Barut and D. E. Zwanziger, Phys. Rev. **127**, 974 (1952); and J. R. Taylor, *ibid.* **127**, 2257 (1962).

[7] V. N. Gribov and I. Ya. Pomeranchuk, Phys. Rev. Letters **9**, 238 (1962).

[8] See, for example, R. Blankenbecler, M. L. Goldberger, N. N. Khuri, and S. B. Treiman, Ann. Phys. (N. Y.) **10**, 67 (1960).

[9] T. Regge, Nuovo Cimento **18**, 947 (1960).